Linear Dynamical System w. Gaussian noise

$$x_{k+1} = A \cdot x_k + B u_k + \varepsilon \qquad y_k = \underbrace{C x_k}_{\in \mathbb{R}^d} + V$$

$y_k \in \mathbb{R}^p$

Given $(\mu_{k|k}, \Sigma_{k|k})$

① Propagation:
$$\mu_{k+1|k} = A \mu_{k|k} + B u_k$$
$$\Sigma_{k+1|k} = A \Sigma_{k|k} A^T + R$$

② Update:
$$\mu_{k+1|k+1} = \mu_{k+1|k} + K_{k+1} (y_{k+1} - C \mu_{k+1|k})$$
$$\Sigma_{k+1|k+1} = (\Sigma_{k+1|k}^{-1} + C^T Q^{-1} C)^{-1}$$
$$K_{k+1} = \Sigma_{k+1|k} C^T (C \Sigma_{k+1|k} C^T + Q)^{-1}$$

In gen:
$$\begin{bmatrix} x_{k+1} = f(x_k, u_k) + \varepsilon_k \\ y_k = g(x_k) + V \end{bmatrix} \quad [D = 0]$$

$$p(x_k \mid y_1 \cdots y_k)$$

Linearization

Let $y = f(x)$ $\quad x \in \mathbb{R}^d \quad y \in \mathbb{R}^p$

$$\underbrace{y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{p=2} = f\underbrace{\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)}_{d=3} \Rightarrow \underbrace{\begin{bmatrix} x_1^2 + x_2 x_3 \\ \sin x_2 + \cos x_3 \end{bmatrix}}_{} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$$

$$\underset{J(x)}{\frac{df(x)}{dx}} = \begin{bmatrix} 2x_1 & x_3 & x_2 \\ 0 & \cos x_2 & -\sin x_3 \end{bmatrix}_{p \times d} = \text{Jacobian}$$

$x=a$ = $J(a)$

$$y = f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_p(x) \end{bmatrix} \qquad \nabla f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial f_1(x)}{\partial x_1} & & \frac{\partial f_p(x)}{\partial x_d} \end{bmatrix}$$

$$\underset{(x=a)}{f(x)} \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$x \in \mathbb{R}^d$

$$f(x) \approx f(a) + \underbrace{\left.\frac{\partial f(x)}{\partial x}\right|_{(x=a)}}_{}(x-a) + \underbrace{\text{h.o.t.}}_{\text{Taylor Series}}$$

$$x \in \mathbb{R}^d \equiv \mathcal{N}(\mu_x, \Sigma_x) \qquad \text{if } y = Ax$$

$$E(y) = E(Ax)$$

$$= A \cdot \mu_x$$

$$y = f(x)$$

$$E(y) = ?$$

$$y = f(x) \approx f(\mu_x) + \underbrace{\frac{\partial f(x)}{\partial x}\bigg|_{x=\mu_x}}_{J(\mu_x)} (x - \mu_x)$$

$$y \approx Jx + f(\mu_x) - J\mu_x$$

$$E\left(\underbrace{Jx} + \underbrace{f(\mu_x)} - \underbrace{J\mu_x}\right)$$

$$E(y) \approx \cancel{J\mu_x} + f(\mu_x) - \cancel{J\mu_x}$$

$$\approx f(\mu_x)$$

Now, non-lin dyn sys w. Gaussian noise

$$x_{k+1} = f(x_k, u_k) + \varepsilon_k$$

$$y_k = g(x_k) + v_k \qquad [D \neq 0]$$

Assume $(\mu_{k|k}, \Sigma_{k|k})$

1 Given

$$(\mu_{k|k}, \Sigma_{k|k}) \xrightarrow{P} (\mu_{k+1|k}, \Sigma_{k+1|k})$$

(1) $$x_{k+1} = f(x_k, u_k) + \varepsilon_k$$

$$x_{k+1} \approx \underline{f(\mu_{k|k}, u_k)} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=\mu_{k|k}}}_{A} (x - \mu_{k|k}) + \varepsilon_k$$

$A = J(\mu_{k|k})$

$$x_{k+1} \approx Ax + \underline{f(\mu_{k|k}, u_k)} - A \cdot \mu_{k|k} + \underline{\varepsilon_k}$$

$$E(x_{k+1}) = \mu_{k+1|k} = f(\mu_{k|k}, u_k)$$

$$\Sigma_{k+1|k} \approx A \Sigma_{k|k} A^T + R$$

(1) $\rightarrow$ $$\mu_{k+1|k} = f(\mu_{k|k}, u_k)$$

$$\Sigma_{k+1|k} \approx A \Sigma_{k|k} A^T + R$$

Step 2 : Update

Goal: $\left(\mu_{k+1|k}, \Sigma_{k+1|k}\right) \to \left(\mu_{k+1|k+1}, \Sigma_{k+1|k+1}\right)$

$$y_{k+1} = g(x_{k+1}) + v_{k+1}$$

$$\approx g(\mu_{k+1|k}) + \underbrace{\frac{\partial g}{\partial x}\Big|_{x=\mu_{k+1|k}}}_{C = J(\mu_{k+1|k})} (x_{k+1} - \mu_{k+1|k}) + v_{k+1}$$

$$y_{k+1} \approx C x_{k+1} + g(\mu_{k+1|k}) - C\mu_{k+1|k} + v_{k+1} \leftarrow$$

dummy observation:

$$y'_{k+1} = y_{k+1} - g(\mu_{k+1|k}) + C \cdot \mu_{k+1|k}$$

$$y'_{k+1} \approx C x_{k+1} + v_{k+1} \leftarrow \text{Lin Ob eq}$$

$\left(\mu_{k+1|k}, \Sigma_{k+1|k}\right)$ from Step 1

Compute $\left(\mu_{k+1|k+1}, \Sigma_{k+1|k+1}\right)$ Step 2 Update of KF

$$\mu_{k+1|k+1} = \mu_{k+1|k} + K_{k+1}\left(\underline{y'_{k+1}} - C\,\mu_{k+1|k}\right)$$

$$\mu_{k+1|k+1} = \mu_{k+1|k} + K\left(y_{k+1} - g\left(\mu_{k+1|k}\right)\right)$$

$$\Sigma_{k+1|k+1} = \left(\Sigma_{k+1|k}^{-1} + C^{T} Q^{-1} C\right)^{-1}$$