

# 1 Chapter 2

## 2 Introduction to State 3 Estimation

### Reading

1. Barfoot, Chapter 2.1-2.2
2. Thrun, Chapter 2
3. Russell Chapter 15.1-15.3

### 4 2.1 A review of probability

5 Probability is a very useful construct to reason about real systems which we  
6 cannot model fully at all scales. It is a fundamental part of robotics. No matter  
7 how sophisticated your camera, it will have noise in how it measures the real  
8 world around it. No matter how good your model for a motor is, there will be  
9 modeled effects which make it move a little differently than how you would  
10 expect. We begin with a quick review of probability, you can read more at  
11 many sources, e.g., [MIT's OCW](#).

12 An experiment is a procedure which can be repeated infinitely and has a  
13 well-defined set of possible outcomes, e.g., the toss of a coin or the roll of dice.  
14 The outcome itself need not always be deterministic, e.g., depending upon  
15 your experiment, the coin may come up heads or tails. We call the set  $\Omega$  the  
16 *sample space*, it is the set of all possible outcomes of an experiment. For two  
17 coins, this set would be

$$\Omega = \{HH, HT, TH, TT\}.$$

18 We want to pick this set to be right granularity to answer relevant questions,  
19 e.g., it is correct but not useful for  $\Omega$  to be position of all the molecules in the

1 coin. After every experiment, in this case tossing the two coins once each, we  
2 obtain an event, it is a subset *event*  $A \subset \Omega$  from the sample space.

$$A = \{HH\}.$$

3 Probability theory is a mathematical framework that allows us to reason  
4 about phenomena or experiments whose outcome is uncertain. Probability of  
5 an event

$$P(A)$$

6 is a function that maps each event  $A$  to a number between 0 and 1: closer to 1  
7 this number, stronger our belief that the outcome of the experiment is going to  
8 be  $A$ .

9 **Axioms** Probability is formalized using a set of three basic axioms that are  
10 intuitive and yet very powerful, they are known a Kolmogorov's axioms. They  
11 are

- 12 • Non-negativity:  $P(A) \geq 0$
- 13 • Normalization:  $P(\Omega) = 1$
- 14 • Additivity: If two events  $A, B$  are such that  $A \cap B = \emptyset$ , then

$$P(A \cup B) = P(A) + P(B).$$

15 You can use these axioms to say things like  $P(\emptyset) = 0$ ,  $P(A^c) = 1 - P(A)$ , or  
16 if  $A \subseteq B$  then  $P(A) \leq P(B)$ .

17 **Conditioning on events** Conditioning helps us answer questions like

$$P(A | B) := \text{probability of } A \text{ given that } B \text{ occurred.}$$

18 Effectively, the sample space has now shrunk from  $\Omega$  to the event  $B$ . It would  
19 be silly to have a null sample-space, so let's say that  $P(B) \neq 0$ . We define  
20 conditional probability as

$$P(A | B) = \frac{P(A \cap B)}{P(B)}; \quad (2.1)$$

21 the probability is undefined if  $P(B) = 0$ . Using this definition, we can  
22 compute the probability of events like "what is the probability of rolling a 2  
23 on a die given that an even number was rolled".

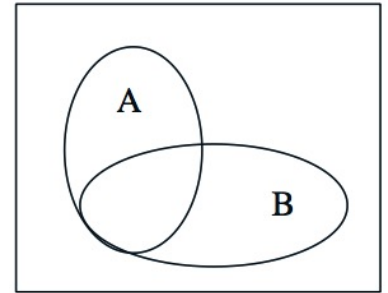
24 We can use this trick to get the law of total probability: if a finite number  
25 of events  $\{A_i\}$  form a partition of  $\Omega$ , i.e.,

$$A_i \cap A_j = \emptyset \quad \forall i, j, \text{ and } \bigcup_i A_i = \Omega$$

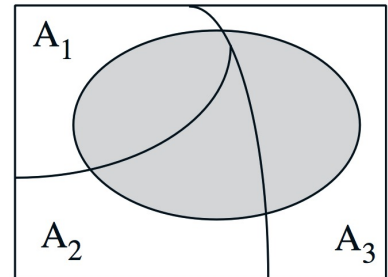
$$26 \quad P(B) = \sum_i P(B | A_i) P(A_i). \quad (2.2)$$

❗ What is an Axiom?

❗



❗ Partitioning the sample space



1 **Bayes' rule** Imagine that instead of someone telling us that the conditioning  
 2 event actually happened, we simply had a belief

$$P(A_i)$$

3 about the possibility of such events  $\{A_i\}$ . For each of  $A_i$ , we can compute the  
 4 conditional probability  $P(B | A_i)$  using (2.1). Say we run our experiment and  
 5 observe that  $B$  occurred, how would our belief on the events  $A_i$  change? In  
 6 other words, we wish to compute

$$P(A_i | B).$$

7 This is the subject of Bayes' rule.

$$\begin{aligned} P(A_i | B) &= \frac{P(A_i \cap B)}{P(B)} \\ &= \frac{P(A_i) P(B|A_i)}{P(B)} \\ &= \frac{P(A_i) P(B|A_i)}{\sum_j P(A_j) P(B | A_j)}. \end{aligned} \tag{2.3}$$

8 The Bayes' rule naturally leads to the concept of independent events. Two  
 9 events  $A, B \subset \Omega$  are independent if observing one does not give us any  
 10 information about the other

$$P(A \cap B) = P(A) P(B). \tag{2.4}$$

11 This is different from disjoint events. Disjoint events never co-occur, i.e.,  
 12 observing one tells us that the other one *did not* occur.

13 **Probability for experiments with real-valued outcomes** We need some  
 14 more work in defining probability for events with real-valued outcomes. The  
 15 sample space is easy enough to understand, e.g.,  $\Omega = [0, 1]$  for your score  
 16 at the end of this course. We however run into difficulties if we define the  
 17 probability of general subsets of  $\Omega$  in terms of the probabilities of elementary  
 18 outcomes (*elements* of  $\Omega$ ). For instance, if we wish to model all elements  
 19  $\omega \in \Omega$  to be equally likely, we are forced to assign each element  $\omega$  a probability  
 20 of zero (to be consistent with the second axiom of probability). This is not very  
 21 helpful in determining the probability of the score being 0.9. If you instead  
 22 assigned some small non-zero number to  $P(\omega_i)$ , then we have undesirable  
 23 conclusions such as

$$P(\{1, 1/2, 1/3, \dots\}) = \infty.$$

24 The way to fix this is to avoid defining the probability of a set in terms  
 25 of the probability of elementary outcomes and work with more general sets.  
 26 While we would ideally like to be able to specify the probability of every  
 27 subset of  $\Omega$ , it turns out that we cannot do so in a mathematically consistent  
 28 way. The trick then is to work with a smaller object known as a  $\sigma$ -algebra, that  
 29 is the set of "nice" subsets of  $\Omega$ .

Given a sample space  $\Omega$ , a  $\sigma$ -algebra  $\mathcal{F}$  (also called a  $\sigma$ -field) is a collection of subsets of  $\Omega$  such that

- $\emptyset \in \mathcal{F}$
- If  $A \in \mathcal{F}$ , then  $A^c \in \mathcal{F}$ .
- If  $A_i \in \mathcal{F}$  for every  $i \in \mathbb{N}$ , then  $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$ .

In short,  $\sigma$ -algebra is a collection of subsets of  $\Omega$  that is closed under complement and countable unions. The pair  $(\Omega, \mathcal{F})$ , also called a measurable space, is now used to define probability of events. A set  $A$  that belongs to  $\mathcal{F}$  is called an event. The probability measure

$$P : \mathcal{F} \rightarrow [0, 1].$$

assigns a probability to events in  $\mathcal{F}$ . We cannot take  $\mathcal{F}$  to be too small, e.g., elements of  $\mathcal{F} = \{\emptyset, \Omega\}$  are easy to construct our  $P$  but are not very useful. For technical reasons, the  $\sigma$ -algebra cannot be too large; notice that we used this concept to *avoid* considering every subset of the sample space  $\mathcal{F} = 2^\Omega$ . Modern probability is defined using a Borel  $\sigma$ -algebra. Roughly speaking, this is an  $\mathcal{F}$  that is just large enough to do interesting things but small enough that mathematical technicalities do not occur.

### 2.1.1 Random variables

A random variable is an assignment of a value to every possible outcome. Mathematically, in our new language of a measurable space, a random variable is a function

$$X : \Omega \rightarrow \mathbb{R}$$

if the set  $\{\omega : X(\omega) \leq c\}$  is  $\mathcal{F}$ -measurable for every number  $c \in \mathbb{R}$ . This is equivalent to saying that every preimage of the Borel  $\sigma$ -algebra on reals  $\mathcal{B}(\mathbb{R})$  is in  $\mathcal{F}$ . A statement  $X(\omega) = x = 5$  means that the outcome of our experiment happens to  $\omega \in \Omega$  when the realized value of the random variable is a particular number  $x$  equal to 5 in our case.

We can now define functions of random variables, e.g., if  $X$  is a random variable, the function  $Y = X^3(\omega)$  for every  $\omega \in \Omega$ , or  $Y = X^3$  for short, is a new random variable. An indicator random variable is a special. If  $A \subset \Omega$ , let  $I_A : \Omega \rightarrow \{0, 1\}$  be the indicator function of this set  $A$ , i.e.,  $I_A(\omega) = 1$  if  $\omega \in A$  and zero otherwise. If our set  $A \in \mathcal{F}$ , then  $I_A$  is an indicator random variable.

**Probability mass functions** The probability law, or a probability distribution, of a random variable  $X$  is denoted by

$$p_X(x) := P(X = x) = P(\{\omega \in \Omega : X(\omega) = x\}).$$

We denote probability distribution using a lower-case  $p$ . It is a function of the realized value  $x$  in the range of a random variable, and  $p_X(x) \geq 0$  (the probability is non-zero) and  $\sum_x p_X(x) = 1$  if  $X$  takes on a discrete number of values. For instance, if  $X$  is the number of coin tosses until the first head, if

❶ Random variables are typically denoted using capital letters,  $X, Y, Z$  although we will be sloppy and not always do so in this course to avoid complicated notation. The distinction between a random variable and the value that it takes will be clear from context.

❷ Let us check that  $Y$  satisfies our definition of a random variable. If  $\{\omega : X(\omega) \leq c\}$  lies in  $\mathcal{F}$  then the set  $\{\omega : Y(\omega) \leq c^{1/3}\}$  also lies in  $\mathcal{F}$ .

❸ The function  $I_A$  is not a random variable if  $A \notin \mathcal{F}$ , but this is, as we said in the previous section, a mathematical corner case. Most subset of  $\Omega$  belong to  $\mathcal{F}$ .

1 we assume that our tosses are independent  $P(H) = p > 0$ , then we have

$$p_X(k) = P(X = k) = P(TT \cdots TH) = (1 - p)^{k-1} p$$

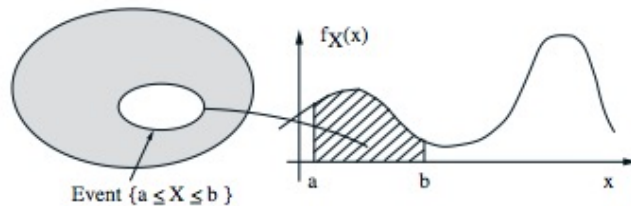
2 for all  $k = 1, 2, \dots$ . This is what is called a geometric probability mass  
3 function.

4 **Cummulative distribution function** A cummulative distribution function  
5 (CDF) is the probability of a random variable  $X$  taking a value less than an  
6 particular  $x \in \mathbb{R}$ , i.e.,

$$F_X(x) = P(X \leq x).$$

7 The CDF  $F_X(x)$  is a non-decreasing function of  $x$ . It converges to zero as  
8  $x \rightarrow -\infty$  and goes to 1 as  $x \rightarrow \infty$ .

9 **Probability density functions** A continuous random variable, i.e., one that  
10 takes values in  $\mathbb{R}$  is described by a probability density function.



11

12 If  $F_X(x)$  is the CDF of an r.v.  $X$  and  $X$  takes values in  $\mathbb{R}$ , the probability  
13 density function (PDF)  $f_X(x)$  (or sometimes also denoted by  $p_X(x)$ ) is defined  
14 to be

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

15 We also have the following relationship between the CDF and the PDF, the  
16 former is the integral of the latter:

$$P(-\infty \leq X \leq x) = F_X(x) = \int_{-\infty}^x f_X(x) dx.$$

17 This leads to the following interpretation of the probability density function:

$$P(x \leq X \leq x + \delta) \approx f_X(x) \delta.$$

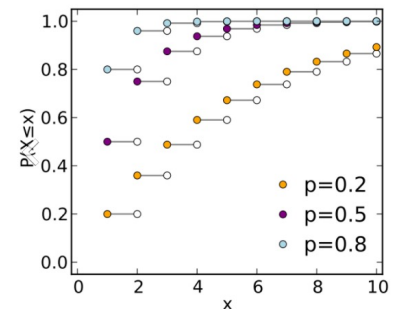
18 **Expectation and Variance** The expected value of a random variable  $X$  is

$$E[X] = \sum_x x p_X(x)$$

19 and denotes the center of gravity of the probability mass function. Roughly  
20 speaking, it is the average of a large number of repetitions of the same experi-  
21 ment. Expectation is a linear, i.e.,

$$E[aX + b] = aE[X] + b$$

❶ The CDF of a geometric random variable for different values of  $p$



Note that CDFs need not be continuous, in the case of a geometric r.v. since the values that  $X$  takes belong to the set of integers, the CDF is constant between any two integers.

1 for any constants  $a, b$ . For two independent random variables  $X, Y$  we have

$$E[XY] = E[X] E[Y].$$

2 We can also compute the expected value of any function  $g(X)$  using the  
3 same formula

$$E[g(X)] = \sum_x g(x) p_X(x).$$

4 In particular, if  $g(x) = x^2$  we have the second moment  $E[X^2]$ . The variance  
5 is defined to be

$$\begin{aligned} \text{var}(X) &= E[(X - E[X])^2] \\ &= \sum_x (x - E[X])^2 p_X(x) \\ &= E[X^2] - (E[X])^2. \end{aligned}$$

6 The variance is always non-negative  $\text{var}(X) \geq 0$ . For an affine function of  $X$ ,  
7 we have

$$\text{var}(aX + b) = a^2 \text{var}(X).$$

8 For continuous-valued random variables, the expectation is defined as

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx;$$

9 the definition of the variance remains the same.

10 **Joint distributions** We often wish to think of the joint probability distribu-  
11 tion of multiple random variables, say the location of an autonomous car in all  
12 three dimensions. The cumulative distribution function associated with this  
13 is therefore

$$F_{X,Y,Z}(x, y, z) = P(X \leq x, Y \leq y, Z \leq z).$$

14 Just like we have the probability density of a single random variable, we  
15 can also write the joint probability density of multiple random variables  
16  $f_{X,Y,Z}(x, y, z)$ . In this case we have

$$F_{X,Y,Z}(x, y, z) = \int_{-\infty}^x \int_{-\infty}^y \int_{-\infty}^z f_{X,Y,Z}(x, y, z) dz dy dx.$$

17 The joint probability density factorizes if two random variables are independ-  
18 ent:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \quad \text{for all } x, y.$$

19 Two random variables are uncorrelated if and only if

$$E[XY] = E[X] E[Y].$$

20 Note that independence implies uncorrelatedness, they are not equivalent. The  
21 covariance is defined as

$$\text{cov}(X, Y) = E[XY] - E[X] E[Y].$$

1 **Conditioning** As we saw before, for a single random variable  $X$  we have

$$P(x \leq X \leq x + \delta) \approx f_X(x) \delta.$$

2 For two random variables, by analogy we would like

$$P(x \leq X \leq x + \delta | Y \approx y) \approx f_{X|Y}(x | y) \delta.$$

3 The conditional probability density function of  $X$  given  $Y$  is defined to be

$$f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0.$$

4 For any given  $y$ , the conditional PDF is a normalized section of the joint PDF,  
5 as shown below.

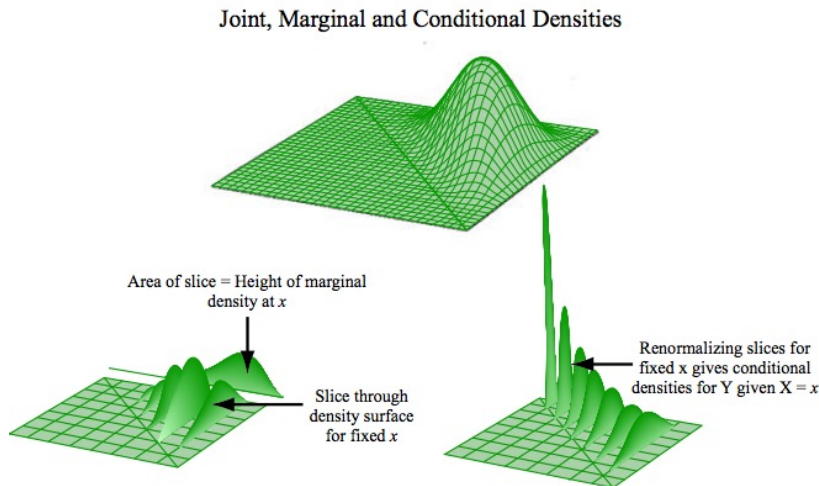


Image by MIT OpenCourseWare, adapted from  
**Probability**, by J. Pittman, 1999.

6  
7 **Continuous form of Bayes rule** We can show using the definition of condi-  
8 tional probability that

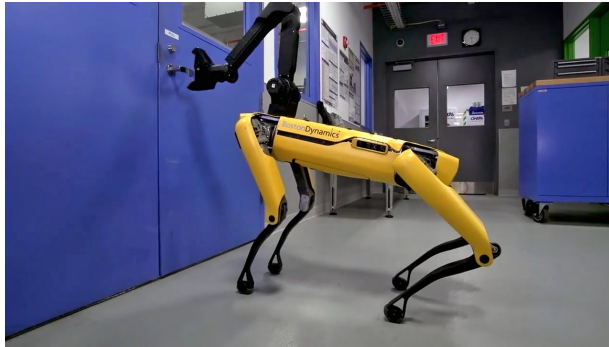
$$f_{Y|X}(y | x) = \frac{f_{X|Y}(x | y) f_Y(y)}{f_X(x)}. \quad (2.5)$$

9 Similarly we also have the law of total probability in the continuous form

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x | y) f_Y(y) dy.$$

## 10 2.2 Using Bayes rule for combining evidence

11 We now study a prototypical state estimation problem. Let us consider a  
12 robot that is trying to check whether the door to a room is open or not.



1

2 We will abstract each observation by the sensors of the robot as a random  
 3 variable  $Y$ . This could be the image from its camera after running some  
 4 algorithm to check the state of the door, the reading from a laser sensor (if  
 5 the time-of-flight of the laser is very large then the door is open), or any  
 6 other mechanism. Let us first distinguish between the cause and effect of the  
 7 observation: if the door is open, we get an observation of a certain kind (laser  
 8 sensor says that the time of flight is large), of course the observation is not  
 9 *caused* by the open door. In other words, we have two kinds of conditional  
 10 probabilities in this problem

$P(\text{open} | Y)$  is a diagnostic quantity, while

$P(Y | \text{open})$  is a causal quantity.

11 Imagine how you would calibrate the sensor in a lab: for each value of the  
 12 state of the door open, not open you would record all the different observations  
 13 received  $Y$  and calculate the conditional probabilities. The causal probability  
 14 is much easier to calculate in this context, one may even use some knowledge  
 15 of elementary physics to model the probability  $P(Y | \text{open})$ , or one may count  
 16 the number of times the observation is  $Y = y$  for a given state of the door.

17 The Bayes rule allows us to transform causal knowledge into diagnostic  
 18 knowledge

$$P(\text{open} | Y) = \frac{P(Y | \text{open}) P(\text{open})}{P(Y)}.$$

19 Remember that the left hand side (diagnostic) is typically something that  
 20 we desire to calculate. Let us put some numbers in this formula. Let  
 21  $P(Y | \text{open}) = 0.6$  and  $P(Y | \text{not open}) = 0.3$ . We will imagine that the  
 22 door is open or closed with equal probability:  $P(\text{open}) = P(\text{not open}) = 0.5$ .  
 23 We then have

$$\begin{aligned} P(\text{open} | Y) &= \frac{P(Y | \text{open}) P(\text{open})}{P(Y)} \\ &= \frac{P(Y | \text{open}) P(\text{open})}{P(Y | \text{open}) P(\text{open}) + P(Y | \text{not open}) P(\text{not open})} \\ &= \frac{0.6 \times 0.5}{0.6 \times 0.5 + 0.3 \times 0.5} = \frac{2}{3}. \end{aligned}$$

24 Notice something very important, the original (prior) probability of the state  
 25 of the door is was 0.5. If we have a sensor that fires with higher likelihood if



1 the door is open, i.e., if

$$\frac{P(Y \mid \text{open})}{P(Y \mid \text{not open})} > 1$$

2 then the probability of the door being open after receiving an observation  
3 *increases*. If the likelihood were less than 1, then observing a realization of  $Y$   
4 would reduce our estimate of the probability of the door being open.

5 **Combining evidence for Markov observations** Say we updated the prior  
6 probability using our first observation  $Y_1$ , let us take another observation  $Y_2$ .  
7 How can we integrate this new observation? It is again an application of Bayes  
8 rule using two observations, or in general multiple observations  $Y_1, \dots, Y_n$ .  
9 Let us imagine this time that  $X = \text{open}$ .

$$P(X \mid Y_1, \dots, Y_n) = \frac{P(Y_n \mid X, Y_1, \dots, Y_{n-1}) P(X \mid Y_1, \dots, Y_{n-1})}{P(Y_n \mid Y_1, \dots, Y_{n-1})}.$$

10 Let us make the very natural assumption that says that our observations from  
11 the sensor  $Y_1, \dots, Y_n$  are independent given the state of the door  $X$ . This is  
12 known as the Markov assumption.

13 We now have

$$\begin{aligned} P(X \mid Y_1, \dots, Y_n) &= \frac{P(Y_n \mid X) P(X \mid Y_1, \dots, Y_{n-1})}{P(Y_n \mid Y_1, \dots, Y_{n-1})} \\ &= \eta P(Y_n \mid X) P(X \mid Y_1, \dots, Y_{n-1}) \end{aligned}$$

14 where

$$\eta^{-1} = P(Y_n \mid Y_1, \dots, Y_{n-1})$$

15 is the denominator. We can now expand the diagnostic probability on the  
16 right-hand side recursively to get

$$P(X \mid Y_1, \dots, Y_n) = \prod_{i=1}^n \eta_i P(Y_i \mid X) P(X). \quad (2.6)$$

17 where  $\eta_i^{-1} = P(Y_i \mid Y_1, \dots, Y_{i-1})$ .

The calculation in (2.6) is very neat and you should always remember

❗ The denominator in Bayes rule, i.e.,  $P(Y)$  is called the evidence in statistics.

it. Given multiple observations  $Y_1, \dots, Y_n$  of the same quantity  $X$ , we can compute the conditional probability  $P(X | Y_1, \dots, Y_n)$  if we code up two functions to compute

- the causal probability (also called the likelihood of an observation)  $P(Y_i | X)$ , and
- the denominator  $\eta_i^{-1}$ .

Given these two functions, we can use the recursion to update multiple observations. The same basic idea also holds if you have two quantities to estimate, e.g.,  $X_1 = \text{open door}$  and  $X_2 = \text{color of the door}$ . The recursive application of Bayes rule lies at the heart of all state estimation methods.

1 Let us again put some numbers into these formulae, imagine that the  
2 observation  $Y_2$  was taken using a different sensor which now has

$$P(Y_2 | \text{open}) = 0.5 \text{ and } P(Y_2 | \text{not open}) = 0.6.$$

3 We have from our previous calculation that  $P(\text{open} | Y_1) = 2/3$  and

$$\begin{aligned} P(\text{open} | Y_1, Y_2) &= \frac{P(Y_2 | \text{open}) P(\text{open} | Y_1)}{P(Y_2 | \text{open}) P(\text{open} | Y_1) + P(Y_2 | \text{not open}) P(\text{not open} | Y_1)} \\ &= \frac{0.5 \times 2/3}{0.5 \times 2/3 + 0.6 \times 1/3} = \frac{5}{8} = 0.625. \end{aligned}$$

4 Notice in this case that the probability that the door is open has reduced from  
5  $P(\text{open} | Y_1) = 2/3$ .

### 6 2.2.1 Coherence of Bayes rule

7 Would the probability change if we used sensor  $Y_2$  before using  $Y_1$ ? In this  
8 case, the answer to this question is no and you are encouraged to perform this  
9 computation for yourselves. Bayes rule is coherent, it will give the same result  
10 regardless of the order of observations.

11 The order of incorporating observation matters if the state of the world  
12 changes while we make observations, e.g., if we have a sensor that tracks  
13 the location of a car, the car presumably moves in between two observations  
14 and we would get the wrong answer if our question was “is there a car at this  
15 location”.

16 As we motivated in the previous chapter, movement is quite fundamental to  
17 robotics and we are typically concerned with estimating the state of a dynamic  
18 world around us using our observations. We will next study the concept of a  
19 Markov Chain with is a mathematical abstraction for the evolution of the state  
20 of the world.

🔗 Can you think of a situation when the order of incorporating observations matters?

## 2.3 Markov Chains

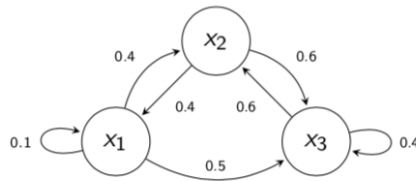
Consider the Whack-The-Mole game: a mole has burrowed a network of three holes  $x_1, x_2, x_3$  into the ground. It keeps going in and out of the holes and we are interested in finding which hole it will show up next so that we can give it a nice whack.

- Three holes:

$$X = \{x_1, x_2, x_3\}.$$

- Transition probabilities:

$$T = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.4 & 0 & 0.6 \\ 0 & 0.6 & 0.4 \end{bmatrix}$$



This is an example of a Markov chain. There is a transition matrix  $T$  which determines the probability  $T_{ij}$  of the mole resurfacing on a given hole  $x_j$  given that it resurfaced at hole  $x_i$  the last time. The matrix  $T^k$  is the  $k$ -step transition matrix

$$T_{ij}^k = P(X_k = x_j \mid X_0 = x_i).$$

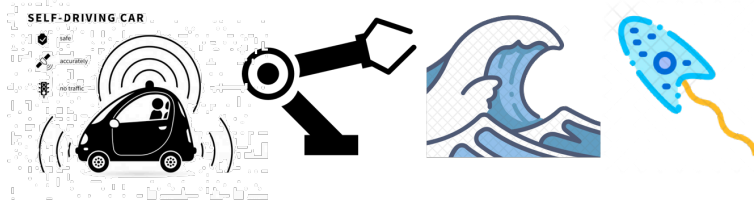
You can see the animations at <https://setosa.io/ev/markov-chains> to build more intuition.

The key property of a Markov chain is that the next state  $X_{k+1}$  is independent of all the past states  $X_1, \dots, X_{k-1}$  given the current state  $X_k$ .

$$X_{k+1} \perp\!\!\!\perp X_1, \dots, X_{k-1} \mid X_k$$

This is known as the Markov property and all systems where we can define a “state” which governs their evolution have this property. Markov chains form a very broad class of systems. For example, all of Newtonian physics fits this assumption.

What is the state of the following systems?



❓ Does a deterministic dynamical system, e.g., a simple pendulum, also satisfy the Markov assumption? What is the transition matrix in this case?

❓ Can you think of a system which does not have the Markov property?

Consider the paramecium above. Its position depends upon a large number of factors: its own motion from the previous time-step but also the viscosity of the material in which it is floating around. One may model the state of the environment around the paramecium as a liquid whose molecules hit thousands of times a second, essentially randomly, and cause disturbances in how the paramecium moves. Let us call this disturbance “noise in the dynamics”. If the

1 motion of the molecules of the liquid has some correlations (does it, usually?),  
 2 this induces correlations in the position of the paramecium. The position of  
 3 the organism is no longer Markov. This example is important to remember,  
 4 the Markov property defined above also implies that the noise in the state  
 5 transition matrix is independent.

6 **Evolution of a Markov chain** The probability of being in a state  $x^i$  at time  
 7  $k + 1$  can be written as

$$P(X_{k+1} = x_i) = \sum_{j=1}^N P(X_{k+1} = x_i \mid X_k = x_j) P(X_k = x_j).$$

8 This equation governs how the probabilities  $P(X_k = x_i)$  change with time  $k$ .  
 9 Let's do the calculations for the Whack-The-Mole example. Say the mole was  
 10 at hole  $x_1$  at the beginning. So the probability distribution of its presence

$$\pi^{(k)} = \begin{bmatrix} P(X_k = x_1) \\ P(X_k = x_2) \\ P(X_k = x_3) \end{bmatrix}$$

11 is such that

$$\pi^{(1)} = [1, 0, 0]^T.$$

12 We can now write the above formula as

$$\pi^{(k+1)} = T' \pi^{(k)} \quad (2.7)$$

13 <sup>1</sup> and compute the distribution  $\pi^{(t)}$  for all times

$$\begin{aligned} \pi^{(2)} &= T' \pi^{(1)} = [0.1, 0.4, 0.5]^T; \\ \pi^{(3)} &= T' \pi^{(2)} = [0.17, 0.34, 0.49]^T; \\ \pi^{(4)} &= T' \pi^{(3)} = [0.153, 0.362, 0.485]^T; \\ &\vdots \\ \pi^{(\infty)} &= \lim_{k \rightarrow \infty} T'^k \pi^{(1)} \\ &= [0.158, 0.355, 0.487]^T. \end{aligned}$$

14 The numbers  $P(X_k = x_i)$  stop changing with time  $k$ . Under certain technical  
 15 conditions, the distribution  $\pi^{(\infty)}$  is unique (single communicating class for a  
 16 Markov chain finite number states). We can compute this invariant distribution  
 17 by writing

$$\pi^{(\infty)} = T' \pi^{(\infty)}.$$

18 We can also compute the distribution  $\pi^{(\infty)}$  directly: the invariant distribution  
 19 is the right-eigenvector of the matrix  $T'$  corresponding to the eigenvalue 1.

20 **Example 2.1.** Consider a Markov chain on two states where the transition

🔍 Do we always know that the transition matrix has an eigenvalue that is 1?

<sup>1</sup>Let us denote the transpose of the matrix  $T$  using the Matlab notation  $T'$  instead of  $T^\top$  for clarity.

1 matrix is given by

$$T = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}.$$

2 The invariant distribution is

$$\begin{aligned} \pi^{(1)} &= 0.5\pi^{(1)} + 0.4\pi^{(2)} \\ \pi^{(2)} &= 0.5\pi^{(1)} + 0.6\pi^{(2)}. \end{aligned}$$

3 Note that the constraint for  $\pi$  being a probability distribution, i.e.,  $\pi^{(1)} + \pi^{(2)} =$   
 4 1 is automatically satisfied by the two equations. We can solve for  $\pi^{(1)}, \pi^{(2)}$   
 5 to get

$$\pi^{(1)} = 4/9 \quad \pi^{(2)} = 5/9.$$

## 6 2.4 Hidden Markov Models (HMMs)

7 <sup>2</sup>

8 Markov chains are a good model for how the state of the world evolves  
 9 with time. We may not always know the exact of these systems and only have  
 10 sensors, e.g., cameras, LiDARs, and radars, to record observations. These  
 11 sensors are typically noisy. So we model the observations as random variables.

12 Hidden Markov Models (HMMs) are an abstraction to reason about ob-  
 13 servations of the state of a Markov chain. An HMM is a sequence of random  
 14 variables  $Y_1, Y_2, \dots, Y_n$  such that the distribution of  $Y_k$  only depends upon on  
 15 the hidden state  $X_k$  of the associated Markov chain.

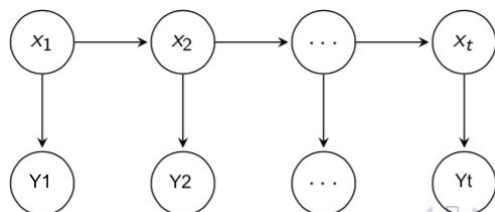


Figure 2.1: A Hidden Markov Model with the underlying Markov chain, the observation at time  $k$  only depends upon the hidden state at that time instant.

16 Notice that an HMM always has an underlying Markov chain behind it.  
 17 For example, if we model the position of a car  $X_k$  as a Markov chain, our  
 18 observation of the position at time  $k$  would be  $Y_k$ . In our example of the  
 19 robot sensing whether the door is open or closed using multiple observations  
 20 across time, the Markov chain is trivial, it is simply the transition matrix  
 21  $P(\text{not open} \mid \text{not open}) = P(\text{open} \mid \text{open}) = 1$ . Just like Markov chains,  
 22 HMMs are a very general class of mathematical models that allow us to think  
 23 about multiple observations across time of a Markov chain.

<sup>2</sup>Parts of this section closely follow Emilio Frazzoli's course notes at [https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-410-principles-of-autonomy-and-decision-making-fall-2010/lecture-notes/MIT16\\_410F10 lec20.pdf](https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-410-principles-of-autonomy-and-decision-making-fall-2010/lecture-notes/MIT16_410F10 lec20.pdf) and [https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-410-principles-of-autonomy-and-decision-making-fall-2010/lecture-notes/MIT16\\_410F10 lec21.pdf](https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-410-principles-of-autonomy-and-decision-making-fall-2010/lecture-notes/MIT16_410F10 lec21.pdf)

Let us imagine that the observations of our HMM are also finite in number, e.g., your score in this course  $\in [0, 100]$  where the associated state of the Markov chain is your expertise in the subject matter. We will write a matrix of observation probabilities

$$M_{ij} = P(Y_k = y_j | X_k = x_i). \quad (2.8)$$

The matrix  $M$  has non-negative entries, after all, each entry is a probability. Since each state has to result in *some* observation, we also have

$$\sum_j M_{ij} = 1.$$

The state transition probabilities of the associated Markov chain are

$$T_{ij} = P(X_{k+1} = x_j | X_k = x_i).$$

Given the abstraction of an HMM, we may be interested in solving a number of problems. We will consider the problem where the state  $X_k$  is the position of a car (which could be stationary or moving) and observations  $Y_k$  give us some estimate of the this position.

1. **Filtering:** Given observations up to time  $k$ , compute the distribution of the state at time  $k$

$$P(X_k | Y_1, \dots, Y_k).$$

This is the most natural problem to understand: we want to find the probability of the car being at a location at time  $k$  given all previous observations. This is a temporally causal prediction, i.e., we are not using any information from the future to reason about the present.

2. **Smoothing:** Given observations up to time  $k$ , compute the distribution of the state at any time  $j < k$

$$P(X_j | Y_1, \dots, Y_k) \quad \text{for } j < k.$$

The observation at a future time  $Y_{k+1}$  gives us some indication of where the car might have been at time  $k$ . In this case we are interested in using the entire set of observations from the past  $Y_1, \dots, Y_j$ , the future  $Y_{j+1}, \dots, Y_k$  to estimate the position of the car. Of course, this problem can only be solved *ex post facto*, i.e., after the time instant  $j$ . An important thing to remember is that we are interested in the position of the car for all  $j < k$  in smoothing.

3. **Prediction:** Given observations up to time  $k$ , compute the distribution of the state at a time  $j > k$

$$P(X_j | Y_1, \dots, Y_k) \quad \text{for } j > k.$$

This is the case when we wish to make predictions about the state of the car  $j > k$  given only observations until time  $k$ . If we knew the underlying Markov chain for the HMM and its transition matrix  $T$ , this would amount to running (2.7) forward using the output of the filtering problem as the initial distribution of the state.

🔍 Why is this true?

- 1 4. **Decoding:** Find the most likely state trajectory  $X_1, \dots, X_k$  that maxi-  
2 mizes the probability

$$P(X_1, \dots, X_k | Y_1, \dots, Y_k)$$

3 given observations  $Y_1, \dots, Y_k$ . Observe that the smoothing problem is  
4 essentially solved independently for all time-steps  $j < k$ . It stands to  
5 reason that if we knew a certain state (say car made a right turn) was  
6 likely given observations at time  $k+1$  and that the traffic light was green  
7 at time  $k$  (given our observations of the traffic light), then we know that  
8 the car did not stop at the intersection at time  $k$ . The decoding problem  
9 allows us to reason about the joint probability of the states and outputs  
10 the most likely trajectory given all observations.

- 11 5. **Likelihood of observations:** Given the observation trajectory,  $Y_1, \dots, Y_k$ ,  
12 compute the probability

$$P(Y_1, \dots, Y_k).$$

13 As you may recall, this is the denominator that we need for the recur-  
14 sive application of Bayes rule. It is made difficult by the fact that we  
15 do not know the state trajectory  $X_1, \dots, X_k$  corresponding to these  
16 observations.

17 These problems are closely related with each other and we will next dig deeper  
18 into them. We will first discuss two building blocks, called the forward and  
19 backward algorithm that together help solve all the above problems.

## 20 2.4.1 The forward algorithm

21 Consider the problem of computing the likelihood of observations. We can  
22 certainly write

$$\begin{aligned} & P(Y_1, \dots, Y_k) \\ &= \sum_{\text{all } (x_1, \dots, x_k)} P(Y_1, \dots, Y_k | X_1, \dots, X_k) P(X_1, \dots, X_k) \\ &= \sum_{\text{all } (x_1, \dots, x_k)} \prod_{i=1}^k P(Y_i = y_i | X_i = x_i) P(X_1 = x_1) \prod_{i=2}^k P(X_i = x_i | X_{i-1} = x_{i-1}) \\ &= \sum_{\text{all } (x_1, \dots, x_k)} M_{x_1 y_1} M_{x_2 y_2} \dots M_{x_k y_k} \pi_{x_1} T_{x_1 x_2} \dots T_{x_{k-1} x_k}. \end{aligned}$$

23 But this is a very large computation, for each possible trajectory  $(x_1, \dots, x_k)$   
24 the states could have taken, we need to perform  $2k$  matrix multiplications.

**Forward algorithm** We can simplify the above computation using the

🔗 How many possible state trajectories are there? What is the total cost of computing the likelihood of observations?

Markov property of the HMM as follows. We will define a quantity known as the forward variable

$$\alpha_k(x) = P(Y_1, \dots, Y_k, X_k = x) \quad (2.9)$$

where  $Y_1, \dots, Y_k$  is our observation sequence up to time  $k$ . Observe now that

1. We can initialize

$$\alpha_1(x) = \pi_x M_{x,y_1} \quad \text{for all } x.$$

2. For each time  $i = 1, \dots, k - 1$ , for all states  $x$ , we can compute

$$\alpha_{k+1}(x) = M_{xy_{k+1}} \sum_{x'} \alpha_k(x') T_{x'x}.$$

using the law of total probability.

3. Finally, we have

$$P(Y_1, \dots, Y_k) = \sum_x \alpha_k(x)$$

by marginalizing over the state variables  $X_k$ .

- 1 This recursion in the forward algorithm is a powerful idea and is much
- 2 faster than our naive summation above.

### 3 2.4.2 The backward algorithm

- 4 Just like the forward algorithm performs the computation recursively in the
- 5 forward direction, we can also perform a backward recursion to obtain the
- 6 probability of the observations. Let us imagine that we have an observation
- 7 trajectory

$$Y_1, \dots, Y_t$$

- 8 up to some time  $t$ . We first define the so-called backward variables which are
- 9 the probability of a future trajectory given the state of the Markov chain at a
- 10 particular time instant

$$\beta_k(x) = P(Y_{k+1}, Y_{k+2}, \dots, Y_t | X_k = x). \quad (2.10)$$

- 11 Notice that the backward variables  $\beta_k$  with the conditioning on  $X_k = x$  are
- 12 slightly different than the forward variables  $\alpha_k$  which are the joint probability
- 13 of the observation trajectory and  $X_k = x$ .

**The Backward algorithm** We can compute the variables  $\beta_k(x)$  recur-

🔗 What is the computational complexity of the Forward algorithm?



sively again as follows.

1. Initialize

$$\beta_t(x) = 1 \quad \text{for all } x.$$

This simply indicates that since we are the end of the trajectory, the future trajectory  $Y_{t+1}, \dots$  does not exist.

2. For all  $k = t - 1, t - 2, \dots, 1$ , for all  $x$ , update

$$\beta_k(x) = \sum_{x'} \beta_{k+1}(x') T_{xx'} M_{x'y_{k+1}}.$$

3. We can now compute

$$P(Y_1, \dots, Y_t) = \sum_x \beta_1(x) \pi_x M_{xy_1}.$$

- 1 **Implementing the forward and backward algorithms in practice** The update equations for both  $\alpha_k$  and  $\beta_k$  can be written using a matrix vector multiplication. We maintain the vectors

$$\begin{aligned} \alpha_k &:= [\alpha_k(x_1), \alpha_k(x_2), \dots, \alpha_k(x_N)] \\ \beta_k &:= [\beta_k(x_1), \beta_k(x_2), \dots, \beta_k(x_N)] \end{aligned}$$

- 4 and can write the updates as

$$\alpha_{k+1}^\top = M_{\cdot, y_{k+1}} \odot (\alpha_k^\top T)$$

- 5 where  $\odot$  denotes the element-wise product and  $M_{\cdot, y_{k+1}}$  is the  $y_{k+1}^{\text{th}}$  column of the matrix  $M$ . The update equation for the backward variables is

$$\beta_k = (\beta_{k+1} \odot M_{\cdot, y_{k+1}}) T.$$

- 7 You must be careful about directly implement these recursions however, because we are iteratively multiplying by matrices  $T, M$  whose entries are all smaller than 1 (they are all probabilities after all), we can quickly run into difficulties where  $\alpha_k, \beta_k$  become too small for some states and we get numerical underflow. You can implement these algorithms in the log-space by writing similar update equations for  $\log \alpha_k$  and  $\log \beta_k$  to avoid such numerical issues.

### 13 2.4.3 Bayes filter

- 14 Let us now use the forward and backward algorithms to solve the filtering problem. We want to compute

$$P(X_k = x \mid Y_1, \dots, Y_k)$$

🔗 What is the computational complexity of running the backward algorithm?

1 for all states  $x$  in the Markov chain. We have that

$$2 \quad \mathbb{P}(X_k = x \mid Y_1, \dots, Y_k) = \frac{\mathbb{P}(X_k = x, Y_1, \dots, Y_k)}{\mathbb{P}(Y_1, \dots, Y_k)} = \eta \alpha_k(x) \quad (2.11)$$

2 where since  $\mathbb{P}(X_k = x \mid Y_1, \dots, Y_k)$  is a legitimate probability distribution on  
3  $x$ , we have

$$\eta = \left( \sum_x \alpha_k(x) \right)^{-1}.$$

4 As simple as that. In order to estimate the state at time  $k$ , we run the forward  
5 algorithm to update variables  $\alpha_i(x)$  from  $i = 1, \dots, k$ . We can implement  
6 this using the matrix-vector multiplication in the previous section.

7 This is a commonly used algorithm known as the Bayes filter and is our  
8 first insight into state estimation.

9 **An important fact** Even if the filtering estimate is computed recursively  
10 using each observation as it arrives, the estimate is actually the probability of  
11 the current state given *all* past observations.

$$\mathbb{P}(X_k = x \mid Y_1, \dots, Y_k) \neq \mathbb{P}(X_k = x \mid Y_k)$$

12 This is an extremely important concept to remember, in state-estimation we  
13 are always interested in computing the state given all available observations.  
14 In the same context, is the following statement true?

$$\mathbb{P}(X_k = x \mid Y_1, \dots, Y_k) = \mathbb{P}(X_k = x \mid Y_k, X_{k-1})$$

## 15 2.4.4 Smoothing

16 For smoothing given observations till time  $t$ , we would like to compute

$$\mathbb{P}(X_k = x \mid Y_1, \dots, Y_t)$$

17 for all time instants  $k = 1, \dots, t$ . Observe the filtering

$$\begin{aligned} \mathbb{P}(X_k = x \mid Y_1, \dots, Y_t) &= \frac{\mathbb{P}(X_k = x, Y_1, \dots, Y_t)}{\mathbb{P}(Y_1, \dots, Y_t)} \\ &= \frac{\mathbb{P}(X_k = x, Y_1, \dots, Y_k, Y_{k+1}, \dots, Y_t)}{\mathbb{P}(Y_1, \dots, Y_t)} \\ &= \frac{\mathbb{P}(Y_{k+1}, \dots, Y_t \mid X_k = x, Y_1, \dots, Y_k) \mathbb{P}(X_k = x, Y_1, \dots, Y_k)}{\mathbb{P}(Y_1, \dots, Y_t)} \\ &= \frac{\mathbb{P}(Y_{k+1}, \dots, Y_t \mid X_k = x) \mathbb{P}(X_k = x, Y_1, \dots, Y_k)}{\mathbb{P}(Y_1, \dots, Y_t)} \\ &= \frac{\beta_k(x) \alpha_k(x)}{\mathbb{P}(Y_1, \dots, Y_t)} \end{aligned} \quad (2.12)$$

18 Study the first step carefully, the numerator is *not* equal to  $\alpha_k(x)$  because  
19 observations go all the way till time  $t$ . The final step uses both the Markov  
20 and the HMM properties: future observations  $Y_{k+1}, \dots, Y_t$  depend only upon

1 future states  $X_{k+1}, \dots, X_t$  (HMM property) which are independent of the  
 2 past observations and states give the current state  $X_k = x$  (Markov property).

3 Smoothing can therefore be implemented by running the forward algorithm  
 4 to update  $\alpha_k$  from  $k = 1, \dots, t$  and the backward algorithm to update  $\beta_k$  from  
 5 time  $k = t, \dots, 1$ .

6 To see an example of smoothing in action, see [ORB-SLAM 2](#). What do  
 7 you think is the state of the Markov chain in this video?

8 **Example for the Whack-the-mole problem** Let us assume that we do not  
 9 see which hole the mole surfaces from (say it is dark outside) but we can hear  
 10 it. Our hearing is not very precise so we have an observation probabilities

$$M = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}.$$

11 Assume that the mole surfaces three times and we make the measurements

$$Y_1 = 1, Y_2 = 3, Y_3 = 3.$$

12 We want to compute the distribution of the states the mole could be in at each  
 13 time. Assume that the we know the mole was in hole 1 at the first step, i.e.,  
 14  $\pi_1 = (1, 0, 0)$  for the Markov chain, like we had in Section 2.3.

15 Run the forward backward algorithm and see that

$$\alpha_1 = (0.6, 0, 0), \alpha_2 = (0.012, 0.048, 0.18), \alpha_3 = (0.0041, 0.0226, 0.0641),$$

16 and

$$\beta_3 = (1, 1, 1), \beta_2 = (0.4, 0.44, 0.36), \beta_1 = (0.1512, 0.1616, 0.1392).$$

17 Using these, we can now compute the filtering and the smoothing state distri-  
 18 butions, let us denote them by  $\pi^f$  and  $\pi^s$  respectively.

$$\pi_1^f = (1, 0, 0), \pi_2^f = (0.05, 0.2, 0.75), \pi_3^f = (0.045, 0.2487, 0.7063)$$

19 and

$$\pi_1^s = (1, 0, 0), \pi_2^s = (0.0529, 0.2328, 0.7143), \pi_3^s = (0.045, 0.2487, 0.7063).$$

20

## 21 2.4.5 Prediction

22 We would like to compute the future probability of the state give observations  
 23 up to some time

$$P(X_k = x | Y_1, \dots, Y_t) \quad \text{for } t < k.$$

24 Here is a typical scenario when you would need this estimate. Imagine that  
 25 you are tracking the position of a car using images from your camera. You are

🔗 Both the filtering problem and the smoothing problem give us the probability of the state given observations. Discuss which one should we should use in practice and why?

🔗 Do you notice any pattern in the solution returned by the filtering and the smoothing problem? Explain why that is the case.

using a deep network to detect the car in each image  $Y_k$  and since the neural network is quite slow, the car moves multiple time steps forward before you get the next observation. As you can appreciate, it would help us compute a more accurate estimate of the conditional probability of  $X_k = x$  if we propagated the position of the car in between successive observations using our Markov chain. This is easy to do.

1. We compute the filtering estimate  $\pi_t^f = P(X_t = x | Y_1, \dots, Y_t)$ , using the forward algorithm.
2. Propagate the Markov chain forward for  $k - t$  time-steps using  $\pi_t^f$  as the initial condition using

$$\pi_{i+1} = T^i \pi_i.$$

## 2.4.6 Decoding: Viterbi's Algorithm

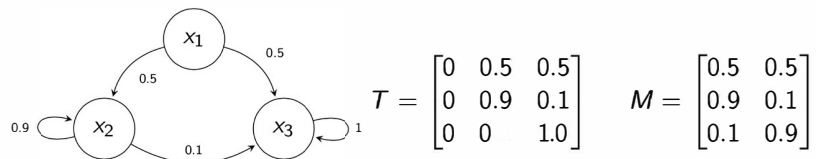
Both filtering and smoothing calculate the probability distribution of the state at time  $k$ . For instance, after recording a few observations, we can compute the probability distribution of the position of the car at each time instant. How do we get most likely trajectory of the car? One option is to choose

$$\hat{X}_k = \operatorname{argmax}_x P(X_k = x | Y_1, \dots, Y_t)$$

at each instant and output

$$(\hat{X}_1, \dots, \hat{X}_t)$$

as the answer. This is however only the point-wise best estimate of the state. This sequence may not be the most likely trajectory of Markov chain underlying our HMM. In the decoding problem, we are interested in computing the most likely state trajectory, not the point-wise most likely sequence of states. Let us take an example of the Whack-the-mole again. We will use a slightly different Markov chain shown below.



There are three states  $x_1, x_2, x_3$  with known initial distribution  $\pi = (1, 0, 0)$  and transition probabilities and observations given by matrices  $T, M$  respectively. Let us say that we only have two observations  $\{y_2, y_3\}$  this time and get the observation sequence

$$(2, 3, 3, 2, 2, 2, 3, 2, 3)$$

from our sensor. The filtering estimates are as follows.

$t$	$x_1$	$x_2$	$x_3$
1	<b>1.0000</b>	0	0
2	0	0.1000	<b>0.9000</b>
3	0	0.0109	<b>0.9891</b>
4	0	0.0817	<b>0.9183</b>
5	0	0.4165	<b>0.5835</b>
6	0	<b>0.8437</b>	0.1563
7	0	0.2595	<b>0.7405</b>
8	0	<b>0.7328</b>	0.2672
9	0	0.1771	<b>0.8229</b>

1

2 The most likely state at each instant is marked in blue. The point-wise most  
3 likely sequence of states is

$$(1, 3, 3, 3, 3, 2, 3, 2, 3).$$

4 Observe that this is not even feasible for the Markov chain. The transition  
5 from  $x_3 \rightarrow x_2$  is not even possible so this answer is clearly wrong. Let us  
6 look at the smoothing estimates.

$t$	$x_1$	$x_2$	$x_3$
1	<b>1.0000</b>	0	0
2	0	<b>0.6297</b>	0.3703
3	0	<b>0.6255</b>	0.3745
4	0	<b>0.6251</b>	0.3749
5	0	<b>0.6218</b>	0.3782
6	0	<b>0.5948</b>	0.4052
7	0	0.3761	<b>0.6239</b>
8	0	0.3543	<b>0.6457</b>
9	0	0.1771	<b>0.8229</b>

7

8 The point-wise most likely states in this case are feasible

$$(1, 2, 2, 2, 2, 2, 3, 3, 3).$$

9 Because the smoothing estimate at time  $k$  also takes into account the observa-  
10 tions from the future  $t > k$ , it effectively eliminates the impossible transition  
11  $x_3 \rightarrow x_2$ . This is still not however the most likely trajectory.

12 We will exploit the Markov property again to calculate the most likely  
13 state trajectory recursively. Let us define the “decoding variables” as

$$\delta_k(x) = \max_{(x_1, \dots, x_{k-1})} \mathbb{P}(X_1 = x_1, \dots, X_{k-1} = x_{k-1}, X_k = x, Y_1, \dots, Y_k); \quad (2.13)$$

14 this is the joint probability of the most likely state trajectory that ends at the  
15 state  $x$  at time  $k$  while generating observations  $Y_1, \dots, Y_k$ . We can now see  
16 that

$$\delta_{k+1}(x) = \max_{x'} \delta_k(x') T_{x'x} M_{x, y_{k+1}}; \quad (2.14)$$

17 the joint probability that the most likely trajectory ends up at state  $x$  at time  
18  $k + 1$  is the maximum of among the joint probabilities that end up at any state

1  $x'$  at time  $k$  multiplied by the one-step state transition  $T_{x'x}$  and observation  
 2  $M_{xy_{k+1}}$  probabilities. We would like to iterate upon this identity to find the  
 3 most likely path. The key idea is to maintain a pointer to the parent state  
 4  $\text{parent}_k(x)$  of the most likely trajectory, i.e., the state from which you could  
 5 have reached  $X_k = x$  given observations. Let us see how.

**Viterbi's algorithm** First initialize

$$\delta_1(x) = \pi_x M_{xy_1}$$

$$\text{parent}_k(x) = \text{null}.$$

for all states  $x$ . For all times  $k = 1, \dots, t - 1$ , for all states  $x$ , update

$$\delta_{k+1}(x) = \max_{x'} \delta_k(x') T_{x'x} M_{x,y_{k+1}}$$

$$\text{parent}_{k+1}(x) = \operatorname{argmax}_{x'} (\delta_k(x') T_{x'x}).$$

The most likely final state is

$$\hat{x}_t = \operatorname{argmax}_{x'} \delta_t(x')$$

and we can now backtrack using our parent pointers to find the most likely trajectory that leads to this state

$$\hat{x}_k = \text{parent}_{k+1}(\hat{x}_{k+1}).$$

The most likely trajectory given observations is

$$\hat{x}_1, \hat{x}_2, \dots, \hat{x}_t$$

and the joint probability of this trajectory and all observations is

$$P(X_1 = \hat{x}_1, \dots, X_t = \hat{x}_t, Y_1 = y_1, \dots, Y_t = y_t) = \delta_t(\hat{x}_t).$$

6 This is a very widely used algorithm, both in robotics and in other areas  
 7 such as speech recognition (given audio, find the most likely sentence spoken  
 8 by the person), wireless transmission and reception, DNA analysis (e.g., the  
 9 state of the Markov chain is the sequence ACTG... and our observations are  
 10 functions of these states at periodic intervals). Its name comes from Andrew  
 11 Viterbi who developed the algorithm in the late 60s, he is one of the founders  
 12 of Qualcomm Inc.

13 Here is how Viterbi's algorithm would look like for our whack-the-mole  
 14 example.

$$\delta_1 = (0.6, 0, 0), \delta_2 = (0.012, 0.048, 0.18), \delta_3 = (0.0038, 0.0216, 0.0432)$$

$$\text{parent}_1 = (\text{null}, \text{null}, \text{null}), \text{parent}_2 = (1, 1, 1), \text{parent}_3 = (2, 3, 3).$$

15 The most likely path is the one that ends in 3 with joint probability 0.0432.

1 This path is (1, 3, 3).

2 Let us also compute Viterbi's algorithm for a longer observation sequence.

$t$	$x_1$	$x_2$	$x_3$
1	0.5/0	0	0
2	0/1	0.025/1	0.225/1
3	0/1	0.00225/2	0.2025/3
4	0/1	0.0018225/2	0.02025/3
5	0/1	0.0014762/2	0.002025/3
6	0/1	0.0011957/2	0.0002025/3
7	0/1	0.00010762/2	0.00018225/3
8	0/1	8.717e-05/2	1.8225e-05/3
9	0/1	7.8453e-06/2	1.6403e-05/3

4 The most likely trajectory is

$$(1, 3, 3, 3, 3, 3, 3, 3, 3).$$

5 Notice that if we had only 8 observations, the most likely trajectory would be

$$(1, 2, 2, 2, 2, 2, 2, 2).$$

6

7 What is the computational complexity of Viterbi's algorithm? It is linear  
 8 in the time-horizon  $t$  and quadratic in the number of states in the Markov  
 9 chain. We are plucking out the most likely trajectory out of  $\text{card}(X)^t$  possible  
 10 trajectories using the  $\delta_k$  variables. Does this remind you of some other problem  
 11 that you may have seen before?

## 12 2.4.7 Shortest path on a Trellis graph

13 You may have seen Dijkstra's algorithm before that computes the shortest path  
 14 to reach a node in the graph given costs of traversing every edge.

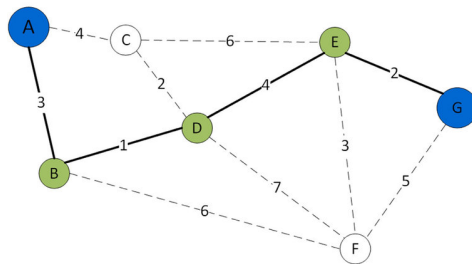


Figure 2.2: A graph with costs assigned to every edge. Dijkstra's algorithm finds the shortest path in this graph between nodes A and B using dynamic programming.

15 In the case of Viterbi's algorithm, we are also interested in finding the

**i** Just like the Bayes filter, Viterbi's algorithm is typically implemented using  $\log \delta_k(x)$  to avoid numerical underflows. This is particularly important for Viterbi's algorithm: since  $\delta_k(x)$  is the probability of an entire state and observation trajectory it can get small very quickly for unlikely states (as seen in the figure).

1 most likely path. For example we can write our joint probabilities as

$$P(X_1, X_2, X_3 | Y_1, Y_2, Y_3) = \frac{P(Y_1 | X_1) P(Y_2 | X_2) P(Y_3 | X_3) P(X_1) P(X_2 | X_1) P(X_3 | X_2)}{P(Y_1, Y_2, Y_3)}.$$

$$\Rightarrow \log P(X_1, X_2, X_3 | Y_1, Y_2, Y_3) = \log P(Y_1 | X_1) + \log P(Y_2 | X_2) + \log P(Y_3 | X_3) \\ + \log P(X_1) + \log P(X_2 | X_1) + \log P(X_3 | X_2) - \log P(Y_1, Y_2, Y_3).$$

2 To find the most likely trajectory, we want to minimize  $-\log P(X_1, X_2, X_3 | Y_1, Y_2, Y_3)$ .

3 The term  $\log P(Y_1, Y_2, Y_3)$  does not depend on  $X_1, X_2, X_3$  and is a constant

4 as far as the most likely path given observations is concerned. We can now

5 write down the “Trellis” graph as shown below.

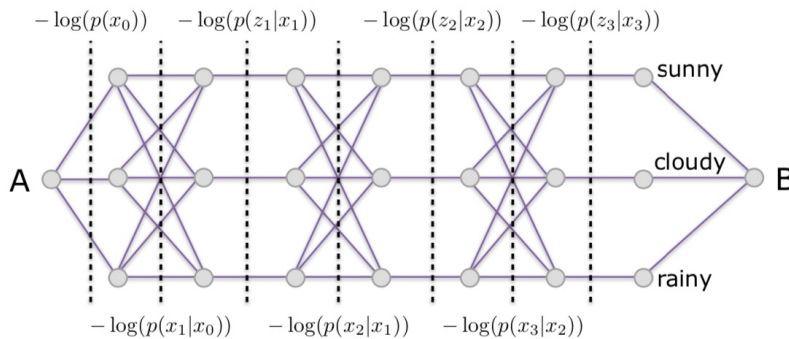


Figure 2.3: A Trellis graph for a 3-state HMM for a sequence of three observations. Disregard the subscript  $x_0$ .

6 Each edge is either the log-probability of the transition of the Markov  
 7 chain, or it is the log-probability of the receiving the observation given a  
 8 state. We create a dummy initial node A and a dummy terminal node B. The  
 9 edge-costs of the final three states, in this case sunny/cloudy/rainy, are zero.  
 10 The costs from node A to the respective states are the log-probabilities of the  
 11 initial state distribution. Dijkstra’s algorithm, which we will study in Module 2  
 12 in more details, now gives the shortest path on the Trellis graph. This approach  
 13 is the same as that of the Viterbi’s algorithm: our parent pointers  $\text{parent}_k(x)$   
 14 are the parent nodes in Dijkstra’s algorithm and our delta variables  $\delta_k(x)$  is the  
 15 cost of each node in the Trellis graph maintained by the Dijkstra’s algorithm.

## 16 2.5 Learning an HMM from observations

17 In the previous sections, given an HMM that had an initial distribution  $\pi$  for the  
 18 Markov chain, a transition matrix  $T$  for the Markov chain and an observation  
 19 matrix  $M$

$$\lambda = (\pi, T, M)$$

20 we computed various quantities such as

$$P(Y_1, \dots, Y_t; \lambda)$$



1 for an observation sequence  $Y_1, \dots, Y_t$  of the HMM. Given an observation  
 2 sequence, we can also go back and update our HMM to make this observation  
 3 sequence more likely. This is the simplest instance of *learning* an HMM. The  
 4 prototypical problem to imagine that our original HMM  $\lambda$  comes from our  
 5 knowledge of the original problem (say a physics model of the dynamics of a  
 6 robot and its sensors). Given more data, namely the observations, we want to  
 7 update this model. The most natural way to update the model is to maximize  
 8 the likelihood of observations given our model, i.e.,

$$\lambda^* = \underset{\lambda}{\operatorname{argmax}} P(Y_1, \dots, Y_t; \lambda).$$

9 This is known as maximum-likelihood estimation (MLE). In this section  
 10 we will look at the Baum-Welch algorithm which solves the MLE problem  
 11 iteratively. Given  $\lambda$ , it finds a new HMM  $\lambda' = (\pi', T', M')$  (the  $'$  denotes a  
 12 new matrix, not the transpose here) such that

$$P(Y_1, \dots, Y_t; \lambda') > P(Y_1, \dots, Y_t; \lambda).$$

13 Let us consider a simple problem. We are going to imagine that the FBI  
 14 is trying to catch the dangerous criminal Keyser Soze who is known to travel  
 15 between two cities Los Angeles (LA) which will be state  $x_1$  and New York  
 16 City (NY) which will be state  $x_2$ . The FBI initially have no clue about his  
 17 whereabouts, so their initial belief on his location is uniform  $\pi = [0.5, 0.5]$ .  
 18 His movements are modeled using a Markov chain

$$T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

19 e.g., if Soze is in LA, he is likely to stay in LA or go to NY with equal  
 20 probability. The FBI can make observations about him, they either observe  
 21 him to be in LA ( $y_1$ ), NY ( $y_2$ ) or do not observe anything at all (null,  $y_3$ ).

$$M = \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}.$$

22 Say that they received an observation sequence of 20 periods

(null, LA, LA, null, NY, null, NY, NY, NY, null, NY, NY, NY, NY, NY, null, null, LA, LA, NY).

23 Can we say something about the probability of Soze's movements? At each  
 24 time  $k$  we can compute

$$\gamma_k(x) := P(X_k = x \mid Y_1, \dots, Y_t)$$

25 the smoothing probability. We can also compute the most likely state trajectory  
 26 he could have take given our observations using decoding. Let us focus on the  
 27 smoothing probabilities  $\gamma_k(x)$  as shown below.

$t$	LA	NY
1	0.5556	0.4444
2	0.8000	0.2000
3	0.8000	0.2000
...	...	...
18	0.8000	0.2000
19	0.8000	0.2000
20	0.1667	0.8333

1

2 The point-wise most likely sequence of states after doing so turns out to be

(LA, LA, LA, LA, NY, LA, NY, NY, NY, LA, NY, NY, NY, NY, NY, LA, LA, LA, LA, NY).

3 Notice how smoothing fills in the missing observations above.

4 **Expected state visitation counts** The next question we should ask is how  
 5 should we update the model  $\lambda$  given this data. We are going to learn the entries  
 6 of the state-transition using

$$T'_{x,x'} = \frac{E[\text{number of transitions from } x \text{ to } x']}{E[\text{number of times the Markov chain was in state } x]}.$$

7 What is the denominator, it is simply the sum of the probabilities that the  
 8 Markov chain was at state  $x$  at time  $1, 2, \dots, t-1$  given our observations, i.e.,

$$E[\text{number of times the Markov chain was in state } x] = \sum_{k=1}^{t-1} \gamma_k(x).$$

9 The numerator is given in a similar fashion. We will define a quantity

$$\begin{aligned} \xi_k(x, x') &:= P(X_k = x, X_{k+1} = x' \mid Y_1, \dots, Y_t) \\ &= \eta \alpha_k(x) T_{x,x'} M_{x',y_{k+1}} \beta_{k+1}(x'); \end{aligned} \quad (2.15)$$

🔗 Derive the expression for  $\xi_k(x, x')$  for yourself.

10 where  $\eta$  is a normalizing constant such that  $\sum_{x,x'} \xi_k(x, x') = 1$ . Observe that  
 11  $\xi_k$  is the joint probability of  $X_k$  and  $X_{k+1}$

$$\begin{aligned} \xi_k(x, x') &= P(X_{k+1} = x' \mid X_k = x, Y_1, \dots, Y_t) \gamma_k(x) \\ &\neq T_{x,x'} \gamma_k(x) \\ &= P(X_{k+1} = x' \mid X_k = x) P(X_k = x \mid Y_1, \dots, Y_t). \end{aligned}$$

12 The expected value of transitioning between states  $x$  and  $x'$  is

$$E[\text{number of transitions from } x \text{ to } x'] = \sum_{k=1}^{t-1} \xi_k(x, x').$$

1 This gives us our new state transition matrix, you will see in the homework  
2 that it comes to be

$$T' = \begin{bmatrix} 0.47023 & 0.52976 \\ 0.35260 & 0.64739 \end{bmatrix}.$$

3 This is a much better informed FBI than the other we had before beginning  
4 the problem where the transition matrix was all 0.5s.

5 **The new initial distribution** What is the new initial distribution for the  
6 HMM? Recall that we are trying to compute the best HMM given the observa-  
7 tions, so if the initial distribution was

$$\pi = P(X_1)$$

8 before receiving any observations from the HMM, it is now

$$\pi' = P(X_1 | Y_1, \dots, Y_t) = \gamma_1(x);$$

9 the smoothing estimate at the first time-step.

10 **Updating the observation matrix** We can use a similar logic at the expected  
11 state visitation counts to write

$$\begin{aligned} M'_{x,y} &= \frac{E[\text{number of times in state } x, \text{ when observation was } y]}{E[\text{number of times the Markov chain was in state } x]} \\ &= \frac{\sum_{k=1}^t \gamma_k(x) \mathbf{1}_{\{y_k=y\}}}{\sum_{k=1}^t \gamma_k(x)}. \end{aligned}$$

12 You will see in your homework problem that this matrix comes up to be

$$M' = \begin{bmatrix} 0.39024 & 0.20325 & 0.40650 \\ 0.06779 & 0.706214 & 0.2259 \end{bmatrix}.$$

13 Notice how the observation probabilities for the unknown state  $y_3$  have gone  
14 down because the Markov chain does not have those states.

15 The ability to start with a rudimentary model of the HMM and update it  
16 using observations is quite revolutionary. Baum et al. proved in the paper  
17 Baum, Leonard E., et al. "A maximization technique occurring in the sta-  
18 tistical analysis of probabilistic functions of Markov chains." The annals of  
19 mathematical statistics 41.1 (1970): 164-171. Discuss the following questions:

- 20 • When do we stop in our iterated application of the Baum-Welch algo-  
21 rithm?
- 22 • Are we always guaranteed to find the same HMM irrespective of our  
23 initial HMM?
- 24 • If our initial HMM  $\lambda$  is the same, are guaranteed to find the same HMM  
25  $\lambda'$  across two different iterations of the Baum-Welch algorithm?
- 26 • How many observations should we use to update the HMM?